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Axial Monogenic Clifford–Padé Approximants

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Monogenic functions on $\mathscr{R}^{m+1}$ which have axial symmetry have been considered recently by F. Sommen (1988, *J. Math. Anal. Appl.* **130**, 110–113). A particular class of these functions which correspond to functions of a complex variable that are Stieltjes functions is considered. Sequences of Clifford Padé approximants to these axial monogenic functions which correspond to Padé approximants for the Stieltjes' series are defined. It is shown that "paradiagonal" sequences of the approximants converge to the original monogenic function in its domain of monogenicity in $\mathscr{R}^{m+1}$

1. INTRODUCTION

A holomorphic function f of a complex variable z will have a Taylor's series expansion,

$$f(z)=\sum_{j=0}^{\infty} f_j z^j, \tag{1.1}$$

which is convergent for all z in the largest disc centring the origin not containing any singularities of f. In that case a good approximation to f, especially close to the origin, may be obtained from a truncation of this series. However, at points close to the boundary or outside the disc such approximants will be unsatisfactory since (1.1) will converge slowly or diverge.

In spite of this the Taylor's series (1.1) may be used to construct sequences of rational functions which give good approximations to f in its whole domain of holomorphicity. They are called Padé approximants [1] and have been investigated in depth during the last two decades and a wide literature exists on their properties and applications [2–5]. Their definition is as follows:

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DEFINITION 1.1. The $[M/N]$ Padé approximant to the function f is the rational function $f_{M/N}$ defined by the condition

$$f_{M/N}(z)\left[\sum_{n=0}^{N} b_n z^n\right] - \sum_{m=0}^{M} a_m z^m = 0, \tag{1.2}$$

where the coefficients $\{a_m\}$, $\{b_n\}$ are defined by the requirement that

$$f(z)\left[\sum_{n=0}^{N} b_n z^n\right] - \sum_{m=0}^{M} a_m z^m = O(z^{N+M+1}) \tag{1.3a}$$

$$b_0 = 1. \tag{1.3b}$$

Thus to determine $f_{M/N}$ it is necessary to know $\{f_j; j=0, 1, ..., N+M\}$; the coefficients in the numerator and denominator of $f_{M/N}$ are then obtained by solving the linear equations

$$\sum_{n=0}^{\mathrm{Min}(N,s)} b_n f_{s-n} = a_s; \qquad 0 \leqslant s \leqslant M \tag{1.4a}$$

$$\sum_{n=0}^{\mathrm{Min}(N,s)} b_n f_{s-n} = 0; \qquad M < s \leqslant M+N. \tag{1.4b}$$

Although numerical results indicate that sequences of the approximants $f_{M/N}$ usually converge to f throughout domains where the latter is holomorphic and also have the property that the poles of the approximant tend to represent the singularities of f, it is in general not possible to prove appropriate mathematical theorems.

However, convergence theorems do exist when the function f is a series of Stieltjes,

$$f(z) \equiv \int_0^\infty \frac{d\phi(u)}{1+uz}, \tag{1.5}$$

where $\phi(u)$ is a bounded non-decreasing function of u taking on infinitely many different values. Such a theorem may be stated in the following form [4], where $D(\delta, R)$ is the domain illustrated in Fig. 1.

THEOREM 1.1. *Let f defined by (1.5) have the formal power series expansion*

$$f(z) = \sum_{j=0}^{\infty} (-)^j c_j z^j \tag{1.6a}$$

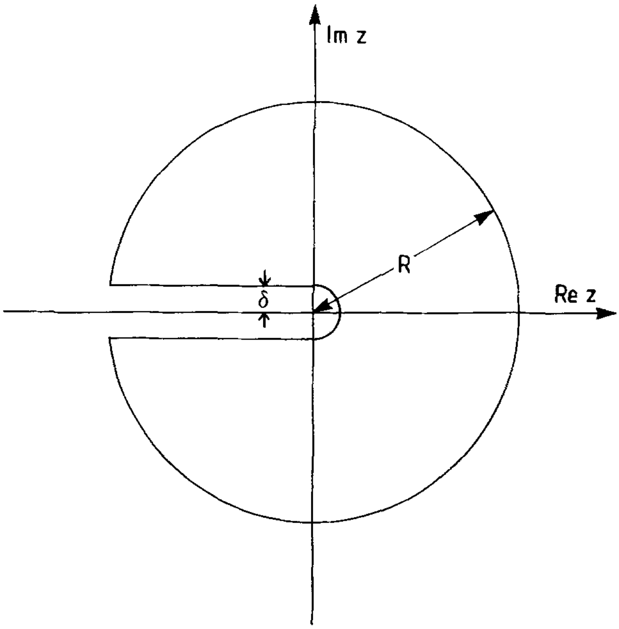

FIG. 1. The domain $D(\delta, R)$ in the complex z-plane.

with

$$c_j \equiv \int_0^\infty u^j \, d\phi(u) \tag{1.6b}$$

such that

$$0 < c_j \lesssim (2j)!. \tag{1.6c}$$

(a) *Then the "paradiagonal" sequences of Padé approximants* $f_{N+j/N}$ *for fixed* $j = -1, 0, 1, \dots$ *converge uniformly on* $D(\delta, R)$ *to* f *as* $N \to \infty$ *for any fixed positive* R, δ.

(b) *The poles of each approximant* $f_{N+j/N}$ *are simple and lie on the negative real axis. Also the residue of an approximant at each of its poles is positive.*

The second part of the theorem implies that the approximants have a partial fraction expansion,

$$f_{N+j/N}(z) = \sum_{n=1}^{N} \frac{\alpha_n}{1+\sigma_n z} + \sum_{l=0}^{j} \beta_l z^l, \qquad N = 1, 2, \dots; j = -1, 0, 1, \dots, \tag{1.7}$$

where α_n, $\sigma_n \geqslant 0$ and β_l are constants (depending on N, j). The second sum is absent when $j = -1$. Comparing (1.7) with the original definition (1.5) of the Stieltjes function, one sees that Padé approximants are obtained by approximating the distribution $d\phi(u)$ by a sum of delta functions with positive weights. This observation has been used to define "generalised" Padé approximants [6, 7].

In the last few decades a hypercomplex function theory has been developed [8–10] which generalises in a natural way the theory of holomorphic functions of one complex variable to functions on the $(m+1)$-dimensional Euclidean space $\mathscr{R}^{m+1}$. These functions take values in the complex Clifford algebra $\mathscr{A}$ defined over $\mathscr{C}^m$. The basis vectors of this algebra are e_k; $k = 1, ..., m$, satisfying the relations

$$e_k e_j + e_j e_k = -2\delta_{jk} 1; \qquad j = 1, ..., m, \tag{1.8}$$

where 1 is the unit element in the algebra which we will denote as e_0. Following [10], we can define $e_A \equiv e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$, where $A \equiv \{\alpha_1, \alpha_2, ..., \alpha_k\}$ is a subset of $\{1, 2, ..., m\}$ with $\alpha_l < \alpha_n$ when $l < n$. Then we can write

$$\mathscr{A} = \left\{ \sum_A a_A e_A ; a_A \in \mathscr{C} \right\}$$

where the sum is over all possible sets A and $e_\phi = e_0 = 1$. We also denote

$$x \equiv x_0 e_0 + \mathbf{x}; \qquad \mathbf{x} \equiv \sum_{k=1}^{m} x_k e_k \tag{1.9}$$

and it corresponds to a point in $\mathscr{R}^{m+1}$. Its norm is defined to be $|x| \equiv [\sum_{i=0}^{m} x_i^2]^{1/2}$.

DEFINITION 1.2. A continuous function f on $\mathscr{R}^{m+1}$ taking values in $\mathscr{A}$ is said to be left monogenic in an open set Ω of $\mathscr{R}^{m+1}$ if

$$(Df)(x) \equiv \sum_{j=0}^{m} e_j \left(\frac{\partial}{\partial x_j} \right) f(x) = 0 \tag{1.10}$$

and right monogenic if

$$(fD)(x) \equiv \sum_{j=0}^{m} \left(\frac{\partial}{\partial x_j} \right) f(x)\, e_j = 0. \tag{1.11}$$

We can write

$$D \equiv \sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j} \equiv e_0 \frac{\partial}{\partial x_0} + D_0, \tag{1.12}$$

where $D_0 \equiv \sum_{j=1}^{m} e_j(\partial/\partial x_j)$ is the Dirac operator in $\mathscr{R}^m$. For the remainder of this paper we will consider left-monogenic functions only, our results being easily extended to the right-monogenic case.

It is easy to see that in the case $m=1$ the monogenic functions become the usual holomorphic functions of complex variable $z=x_1+ix_0$ and (1.10), (1.11) give the usual Cauchy–Riemann equations when we identify e_1 with $-i$.

Also, for $k=1, 2, ..., m$, $Z_k \equiv x_k e_0 - x_0 e_k$ are both left and right monogenic. Polynomial functions of the Z_k's have been constructed which are monogenic and there exist theorems giving expansions of a general monogenic function f in terms of these polynomials [10]. One might think that it would be possible to define Padé type approximants of f taking the form of one truncated sequence of monogenic polynomials divided by another truncated sequence, where the coefficients in the sequences are defined by conditions equivalent to (1.3a), (1.3b) and hence to the linear equations (1.4a, (1.4b).

There are two problems in using such a procedure:

(a) On the LHS of (1.2), (1.3) we have the product $f(z)[\sum_{n=0}^{N} b_n z^n]$ which is holomorphic and has an expansion in powers of z whose coefficients appear on the LHS of (1.4a), (1.4b). However, the straightforward Clifford product of two monogenic functions is not monogenic. Instead one has to take the Cauchy–Kowalewski product [10] which is a symmetrised product of the monogenic polynomials appearing in the expansion of the monogenic functions. Although it appears possible to define a "numerator polynomial" and a "denominator polynomial," the Cauchy–Kowalewski product will appear on the LHS of (1.2). It will therefore not be possible to obtain the approximant by simple division of the numerator by the donominator.

(b) The last remark leads to the second problem in extending Padé approximants to monogenic functions. Clifford algebras are in general not division algebras so in general the "ratio" of two monogenic polynomials will not exist.

To avoid these problems, we will consider the approach of generalising the Stieltjes function $f(z)$ given by (1.5) and its corresponding Padé approximant given as the partial fraction expansion (1.7). To do this we consider axial monogenic functions $f(x)$ discussed recently by Sommen [11]. These functions are defined in axially symmetric domains of $\mathscr{R}^{m+1}$ where the axis of symmetry is in the x_0 direction. In Section 2 we give an introduction to these functions and define axial monogenic power functions. The generalisation of the Stieltjes integral is given in Section 3 and Axial Monogenic Padé approximants corresponding to (1.7) are defined. In Section 4 we discuss the convergence properties of these approximants. We prove that "paradiagonal" sequences of our approximants converge to the original monogenic function in the whole of the monogenicity domain of $f(x)$ in $\mathscr{R}^{m+1}$ and also have a "power matching" property analogous to that for ordinary Padé approximants.

2. Axial Monogenic Functions

Let $\{x_0, \rho, \omega\} \in \mathscr{R} \times \mathscr{R}_+ \times S^{m-1}$ be cylindrical coordinates determined by $x = x_0 e_0 + \rho \boldsymbol{\omega}$ in $\mathscr{R}^{m+1}$, where $\boldsymbol{\omega} \equiv \mathbf{e}_\rho = \mathbf{x}/|\mathbf{x}|$ and $|x| = [x_0^2 + \rho^2]^{1/2}$. Sommen has shown [11] that every left monogenic function f in an open set $\Omega \subseteq \mathscr{R}^{m+1}$ and invariant under $SO(m)$ has an expansion of the form

$$f(x) = \sum_{k=0}^{\infty} f_k(x), \tag{2.1}$$

where f_k is an axial monogenic function of degree k. The f_k may be written in the form

$$f_k(x) = A_{k,\omega}(x_0, \rho) + \mathbf{e}_\rho B_{k,\omega}(x_0, \rho), \tag{2.2}$$

where for (x_0, ρ) fixed, $A_{k,\omega}$ and $B_{k,\omega}$ are inner spherical monogenics of degree k on S^{m-1}. A condition for the f_k's to be monogenic is (cf. F. Sommen, *Rend. Circ. Mat. Palermo* **6** (1984), 259–269)

$$\frac{\partial}{\partial x_0} A_{k,\omega} - \frac{\partial}{\partial \rho} B_{k,\omega} = \frac{k+m-1}{\rho} B_{k,\omega} \tag{2.3a}$$

$$\frac{\partial}{\partial x_0} B_{k,\omega} + \frac{\partial}{\partial \rho} A_{k,\omega} = \frac{k}{\rho} A_{k,\omega}. \tag{2.3b}$$

In the following we will assume that solutions of (2.3a), (2.3b) may be written in the form

$$A_{k,\omega}(x_0, \rho) = A(x_0, \rho)\, P_k(\boldsymbol{\omega}), \tag{2.4a}$$

$$B_{k,\omega}(x_0, \rho) = B(x_0, \rho)\, P_k(\boldsymbol{\omega}), \tag{2.4b}$$

where P_k is an inner spherical monogenic of degree k.

An example of a left axial monogenic function of degree k is

$$F_{k,t}(x) \equiv \frac{[\bar{x} \pm t]}{|x \pm t|^{m+1+2k}} P_k(\mathbf{x}), \tag{2.5}$$

where t is a real parameter and $\bar{x} = x_0 e_0 - \rho \mathbf{e}_\rho$. Using the homogeneity property of the inner spherical monogenics, i.e., $P_k(\mathbf{x}) = \rho^k P_k(\boldsymbol{\omega})$, we see in this case that

$$A(x_0, \rho) = \frac{(x_0 \pm t)\rho^k}{((x_0 \pm t)^2 + \rho^2)^{(m+1+2k)/2}}, \tag{2.6a}$$

$$B(x_0, \rho) = -\frac{\rho^{k+1}}{((x_0 \pm t)^2 + \rho^2)^{(m+1+2k)/2}}. \tag{2.6b}$$

It is straightforward to confirm that the conditions (2.3a), (2.3b) are satisfied so that $F_{k,t}(x)$ is indeed a left axial monogenic function of degree k.

Other examples given by Sommen [11] include generalised power functions:

(a) *Inner Power Functions*

$$p_{s,k,m}(x_0, \rho)\, P_k(\boldsymbol{\omega}) = [A_1(x_0, \rho) + \mathbf{e}_\rho B_1(x_0, \rho)]\, P_k(\boldsymbol{\omega}) \tag{2.7}$$

with

$$A_1(x_0, \rho) = \rho^s F\left(1 - \frac{k+m+s}{2}, \frac{k-s}{2}; \frac{1}{2}; \frac{-x_0^2}{\rho^2}\right), \tag{2.8a}$$

$$B_1(x_0, \rho) = (k-s)\, x_0 \rho^{s-1} F\left(1 - \frac{k+m+s}{2}, \frac{k-s}{2} + 1; \frac{3}{2}; \frac{-x_0^2}{\rho^2}\right). \tag{2.8b}$$

(b) *Outer Power Function*

$$q_{s,k,m}(x_0, \rho)\, P_k(\boldsymbol{\omega}) = [A_2(x_0, \rho) + \mathbf{e}_\rho B_2(x_0, \rho)]\, P_k(\boldsymbol{\omega}) \tag{2.9}$$

with

$$A_2(x_0, \rho) = (s+k+m-1)\, x_0 \rho^{s-1} F\left(\frac{k-s+1}{2}, \frac{3-s-k-m}{2}; \frac{3}{2}; \frac{-x_0^2}{\rho^2}\right), \tag{2.10a}$$

$$B_2(x_0, \rho) = \rho^s F\left(\frac{k-s+1}{2}, \frac{1-s-k-m}{2}; \frac{1}{2}; \frac{-x_0^2}{\rho^2}\right). \tag{2.10b}$$

The nomenclature arises from the fact that when $x_0 = 0$,

$$p_{s,k,m}(x_0, \rho) = \rho^s; \qquad q_{s,k,m}(x_0, \rho) = \mathbf{e}_\rho \rho^s. \tag{2.11}$$

Thus $p_{s,k,m}(x_0, \rho)\, P_k(\boldsymbol{\omega})$ and $q_{s,k,m}(x_0, \rho)\, P_k(\boldsymbol{\omega})$ provide the left monogenic extensions ρ^s and $\mathbf{e}_\rho \rho^s$ from $\mathscr{R}^m$ to the domain $\{x : x_0^2 \leqslant \sum_{l=1}^m x_l^2\}$ in $\mathscr{R}^{m+1}$ where they exist. To simplify the notation, in the remainder of this paper we will omit the indices k, m on these power functions.

We will develop an expansion of $F_{k,t}(x)$ in terms of these generalised power functions. If we take $x_0 = 0$ in (2.5) and consider the upper of the pairs of signs,

$$F_{k,t}(\mathbf{x}) = \frac{[-\rho \mathbf{e}_\rho + t]\, \rho^k P_k(\boldsymbol{\omega})}{[t^2 + \rho^2]^{(m+1+2k)/2}} \tag{2.12}$$

$$= t^{-m-k} \sum_{l=0}^{\infty} \left(1 - \mathbf{e}_\rho \frac{\rho}{t}\right)\left(\frac{\rho}{t}\right)^{2l+k} \frac{(-)^l \Gamma(k + (m+1)/2 + l)\, P_k(\boldsymbol{\omega})}{l!\, \Gamma(k + (m+1)/2)}.$$

We obtain a left monogenic extension to $\mathscr{R}^{m+1}$ by replacing powers of ρ by the corresponding inner power functions and powers of ρ times $\mathbf{e}_\rho$ by outer power functions; then (2.12) becomes, for x in $\mathscr{R}^{m+1}$,

$$F_{k,t}(x) = \frac{[\bar{x} + t]\, P_k(\mathbf{x})}{|x + t|^{m+1+2k}}$$

$$= t^{-2k-m} \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \frac{\Gamma(k + (m+1)/2 + l)}{\Gamma(k + (m+1)/2)} \frac{1}{t^{2l}}$$

$$\times \left[p_{2l+k}(x_0, \rho) - \frac{1}{t} q_{2l+k+1}(x_0, \rho) \right] P_k(\boldsymbol{\omega}). \tag{2.13}$$

We make use of this expansion in the subsequent sections.

3. Axial Monogenic Padé Approximants

We will consider monogenic functions which are generalisations of the series of Stieltjes (1.5). Let

$$f_k(x_0 + \mathbf{x}) \equiv \int_0^\infty \frac{(\bar{x} + t)\, d\phi(t)}{|x + t|^{m+1+2k}} P_k(\mathbf{x}), \tag{3.1a}$$

where $\phi(t)$ is a real bounded non-decreasing function of t taking on infinitely many values or equivalently $d\phi(t)$ is a non-negative real distribution. We will from now on assume $m \geqslant 2$ since $m = 1$ corresponds to the usual functions of a complex variable. Inserting the expansion (2.13) for $F_{k,t}(x)$ defined by (2.5), we see that $f_k(x)$ has the (formal) expansion

$$f_k(x) = \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \left(k + \frac{m+1}{2}\right)_l$$

$$\times [p_{2l+k}(x_0, \rho)\, c_{2l} - q_{2l+k+1}(x_0, \rho)\, c_{2l+1}]\, P_k(\boldsymbol{\omega}), \tag{3.1b}$$

with

$$c_l \equiv \int_0^\infty t^{-l-m-2k}\, d\phi(t);\ (a)_j \equiv \frac{\Gamma(a+j)}{\Gamma(a)}. \tag{3.2}$$

It will be convenient to change the integration variable to $u = 1/t$; then,

$$f_k(x_0 + \mathbf{x}) \equiv \int_0^\infty \frac{[1 + u\bar{x}]\, d\psi(u)}{|1 + ux|^{m+1+2k}} P_k(\mathbf{x}) \tag{3.3}$$

with $d\psi(u) = u^{m+2k}\, d\phi(1/u)$.

To define approximants to $f_k(x)$ we adopt an approach used by Baker [2] to define a generalisation of Padé approximants. We can write (3.3) as

$$f_k(x) \equiv \left[\int_0^\infty [G_1(u, x_0, \rho) - G_2(u, x_0, \rho)\, \mathbf{e}_\rho]\, d\psi(u)\right] P_k(\mathbf{x}), \tag{3.4}$$

where

$$\begin{aligned} G_1(u, x_0, \rho) &= \frac{1 + ux_0}{[(1 + ux_0)^2 + u^2\rho^2]^{(m+1+2k)/2}}; \\ G_2(u, x_0, \rho) &= \frac{u\rho}{[(1 + ux_0)^2 + u^2\rho^2]^{(m+1+2k)/2}} \end{aligned} \tag{3.5}$$

are scalar functions.

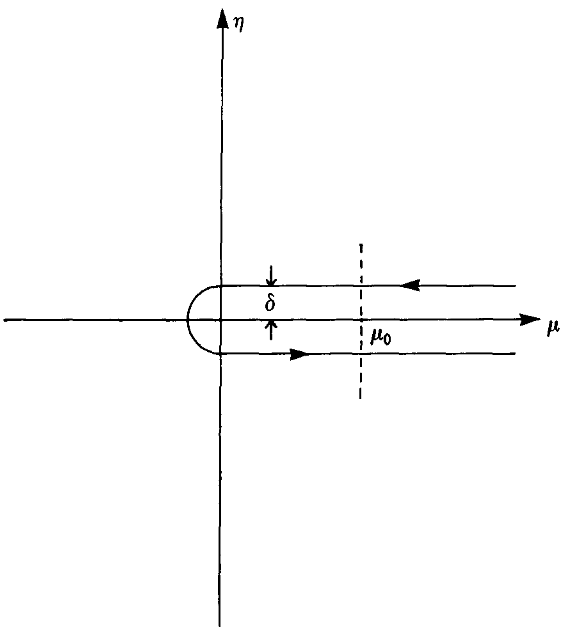

FIG. 2. The contour C enclosing the domain $\Delta(\delta)$ in the complex s-plane.

Let us now consider the G_i's as functions of the complex variable s (replacing u), taking values inside and on the contour C indicated in Fig. 2, i.e., the set $\Delta(\delta) \equiv \{s = \mu + i\eta;\ \mu \geqslant 0,\ |\eta| \leqslant \delta \cup \mu < 0,\ \mu^2 + \eta^2 \leqslant \delta^2\}$. At the same time we consider the domain $\mathscr{D}_m(\lambda, R)$ in $\mathscr{R}^{m+1}$ illustrated in Fig. 3 (for $m = 2$), i.e., the set of points

$$\{x_0^2 + \rho^2 \leqslant R^2;\ x_0 \leqslant 0,\ \rho \geqslant \lambda \cup x_0 > 0,\ \rho^2 + x_0^2 \geqslant \lambda^2\}.$$

It is straightforward to prove that when $R = (4\delta)^{-1/4}$, $\lambda = \delta^{+1/2}$ and $0 < \delta \leqslant \frac{1}{4}$,

$$\mathbf{I}(s, x_0, \rho) \equiv [(1 + sx_0)^2 + s^2\rho^2] \tag{3.6}$$

is never real negative or zero for all $s \in \Delta(\delta)$ and all $x \in \mathscr{D}_m(\lambda, R)$. So for these values of x, $G_i(s, x_0, \rho)$ are holomorphic functions of s on $\Delta(\delta)$.

We use this property to express the scalar and vector parts of the integral in (3.3) as transforms of a Stieltjes integral:

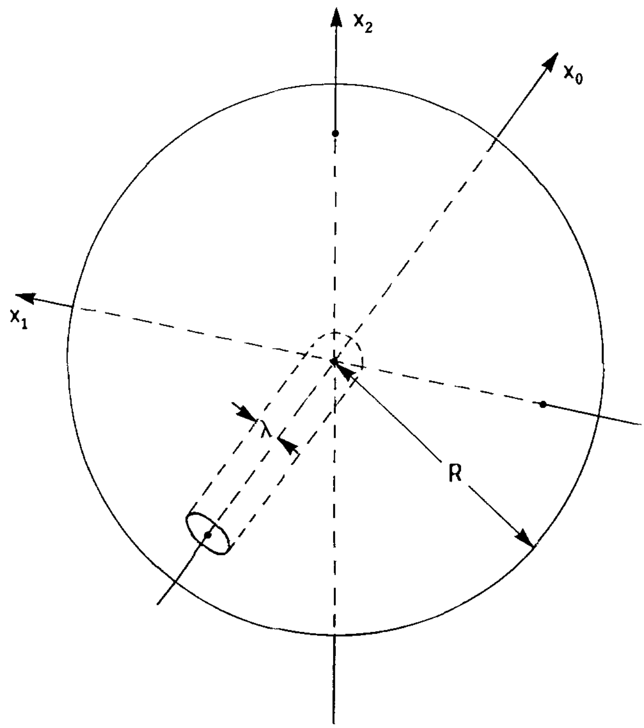

FIG. 3. The domain $\mathscr{D}_m(\lambda, R)$ in $\mathscr{R}_{m+1}$ illustrated for the case $m = 2$.

THEOREM 3.1. *For all x in $\mathscr{D}_m(\lambda, R)$ with R, λ given above and* $0 < \delta \leqslant \frac{1}{4}$,

$$\int_0^\infty G_i(u, x_0, \rho)\, d\psi(u) = \frac{1}{2\pi i} \int_C G_i(s, x_0, \rho)\, H(-1/s) \frac{ds}{s}, \tag{3.7}$$

where the integral over the contour C given in Fig. 2 exists and

$$H(z) = \int_0^\infty \frac{d\psi(u)}{[1 + uz]}. \tag{3.8}$$

Proof. For the given values of R, λ, x, and $s = \mu + i\eta \in C$ it may be proved that

$$|G_i(s, x_0, \rho)| \leqslant a_i(\delta)(|\mu| + 1) \tag{3.9a}$$

and for $\mu \geqslant \delta^{1/4}$,

$$|G_i(s, x_0, \rho)| \leqslant b_i(\delta)\, \mu^{-m-2k}, \tag{3.9b}$$

where $a_i(\delta)$, $b_i(\delta)$ are finite for $0 < \delta \leqslant \frac{1}{4}$. Therefore when $m = 2, 3, ...,$ the RHS of (3.7) exists and is equal to

$$\int_0^\infty \frac{1}{2\pi i} \left[\int_C \frac{G_i(s, x_0, \rho)\, ds}{(s - u)} \right] d\psi(u), \tag{3.10}$$

where the intercharge of order of integration is justified by the use of (3.9). The result then follows from the fact that the $G_i(s, x_0, \rho)$ are holomorphic on $\Delta(\delta)$.

From (3.7) and (3.8) we have for the axial monogenic functions f_k the representation.

$$f_k(x) = \frac{1}{2\pi i} \int_C \frac{[1 + s\bar{x}]}{[(1 + sx_0)^2 + s^2\rho^2]^{(m+1+2k)/2}} H(-1/s) \frac{ds}{s} P_k(\mathbf{x}), \tag{3.11}$$

and this form is used to define Axial Monogenic Padé approximants.

DEFINITION. Let $H_{N+j/N}(z)$; $N = 0, 1, 2, ...$; $j = -1, 0, 1, ...,$ be the $[N + j/N]$ Padé approximant to $H(z)$. Then we define the $[N + j/N]$ Axial Monogenic Padé Approximant of $f_k(x)$ by

$$f^j_{k,N}(x) = \frac{1}{2\pi i} \int_C \frac{(1 + s\bar{x})\, H_{N+j/N}(-1/s)}{[(1 + sx_0)^2 + s^2\rho^2]^{(m+1+2k)/2}} \frac{ds}{s} P_k(\mathbf{x}) \tag{3.12}$$

for all $x \in \mathscr{D}_m(\lambda, R)$.

We will see in (4.13) that these approximants give an "accuracy-through-order" approximation to $f_k(x)$ analogous to that given by Padé approximants to $f(z)$ presented in (1.3a).

A simpler form for these approximants is given by the following:

THEOREM 3.2. *Let the* $[N+j/N]$ *Padé approximant to* $H(z)$ *have the partial fraction expansion corresponding to* (1.7):

$$H_{N+j/N}(z) = \sum_{n=1}^{N} \frac{\alpha_n}{1+\sigma_n z} + \sum_{l=0}^{j} \beta_l z^l. \tag{3.13}$$

Then

$$\begin{aligned} f^j_{k,N}(x) = \Bigg\{ & \left[\sum_{n=1}^{N} \frac{\alpha_n(1+\sigma_n \bar{x})}{|1+\sigma_n x|^{m+1+2k}} \right] \rho^k \\ & + \sum_{r=0}^{[j/2]} \beta_{2r} \frac{(-)^r}{r!} \left(\frac{m+1}{2} + k \right)_r p_{2r+k}(x_0, \rho) \\ & + \sum_{t=0}^{[(j-1)/2]} \beta_{2t+1} \frac{(-)^t}{t!} \left(\frac{m+1}{2} + k \right)_t q_{2t+1+k}(x_0, \rho) \Bigg\} P_k(\boldsymbol{\omega}) \end{aligned} \tag{3.14}$$

for $N = 0, 1, ...;\ j = -1, 0, 1, ...,$ *and as usual the* β *terms are absent when* $j = -1$.

Proof. For series of Stieltjes the σ_n, α_n are real and non-negative so that $H_{N+j/N}(-1/s)$ has poles on the positive real s-axis at $s = \sigma_n$ with $\sigma_n \neq \sigma_m$ for $n \neq m$ and also a pole at $s = 0$. Using the residue theorem

$$\begin{aligned} f^j_{k,N}(x) = & \left\{ \sum_{n=1}^{N} \frac{\alpha_n(1+\sigma_n \bar{x})}{|1+\sigma_n x|^{m+1+2k}} \right\} P_k(\mathbf{x}) \\ & + \frac{1}{2\pi i} \sum_{l=0}^{j} (-)^l \beta_l \int_C \frac{s^{-(l+1)}(1+s\bar{x})\, ds}{[(1+sx_0)^2 + s^2\rho^2]^{[m+1+2k]/2}} P_k(\mathbf{x}). \end{aligned} \tag{3.15}$$

Let us consider the integrals in the second sum on the right (which are absent when $j = -1$). Now

$$g_{l,k}(x) \equiv \frac{1}{2\pi i} \int_C \frac{s^{-(l+1)}(1+s\bar{x})\, ds}{[(1+sx_0)^2 + s^2\rho^2]^{[m+1+2k]/2}} P_k(\mathbf{x}) \tag{3.16}$$

exists for $m = 2, 3, ..., k = 0, 1, 2, ...,$ and $x \in \mathscr{D}_m(\lambda, R)$ with $R^4 = 1/4\delta$, $\lambda = \delta^{+1/2}$, and $0 < \delta \leqslant \frac{1}{4}$ and is a left axial monogenic function in the above domain since the denominator of the integrand does not vanish. It therefore has a unique monogenic extension [10] from the sub-domain $\{x_0=0\}\cap\mathscr{D}_m(\lambda, R)$. For $x_0=0$,

$$g_{l,k}(\mathbf{x})=\frac{1}{2\pi i}\int_C \frac{s^{-(l+1)}(1-s\rho\,\mathbf{e}_\rho)\,ds}{[1+s^2\rho^2]^{(m+1+2k)/2}}\,P_k(\mathbf{x})$$

$$=\left\{\frac{1}{l!}\left[\frac{d^l}{ds^l}\left(\frac{1}{[1+s^2\rho^2]^{(m+1+2k)/2}}\right)\right]_{s=0}\right.$$

$$\left.-\frac{\rho\mathbf{e}_\rho}{(l-1)!}\left[\frac{d^{l-1}}{ds^{l-1}}\frac{1}{[1+s^2\rho^2]^{(m+1+2k)/2}}\right]_{s=0}\right\}P_k(\mathbf{x}) \quad (3.17)$$

since the only singularities of the integrand in $\Delta(\delta)$ are at $s=0$.

But

$$\left[\frac{d^l}{ds^l}\left(\frac{1}{[1+s^2\rho^2]^{(m+1+2k)/2}}\right)\right]_{s=0}$$

$$=0; \qquad l \text{ odd} \qquad (3.18a)$$

$$=\rho^l(-)^{l/2}\frac{l!}{(l/2)!}\left(\frac{m+1}{2}+k\right)_{l/2}; \qquad l\text{ even}, \qquad (3.18b)$$

using the binomial expansion. Therefore

$$g_{l,k}(\mathbf{x})=\frac{(-)^{l/2}}{(l/2)!}\left(\frac{m+1}{2}+k\right)_{l/2}\rho^{l+k}P_k(\boldsymbol{\omega}); \qquad l \text{ even} \qquad (3.19a)$$

$$=\frac{(-)^{(l+1)/2}}{((l-1)/2)!}\left(\frac{m+1}{2}+k\right)_{(l-1)/2}\rho^{l+k}\,\mathbf{e}_\rho P_k(\boldsymbol{\omega}); \qquad l \text{ odd}. \qquad (3.19b)$$

As discussed earlier, the monogenic extensions of $\rho^{l+k}P_k(\boldsymbol{\omega})$ and $\rho^{l+k}\mathbf{e}_\rho P_k(\boldsymbol{\omega})$ are the inner and outer power functions $p_{l+k}(x_0, \rho)\,P_k(\boldsymbol{\omega})$ and $q_{l+k}(x_0, \rho)\,P_k(\boldsymbol{\omega})$, respectively. Therefore when $x_0\neq 0$,

$$g_{l,k}(x)=\frac{(-)^{l/2}}{(l/2)!}\left(\frac{m+1}{2}+k\right)_{l/2}p_{l+k}(x_0, \rho)\,P_k(\boldsymbol{\omega}); \qquad l \text{ even} \quad (3.20a)$$

$$=\frac{(-)^{(l+1)/2}}{((l-1)/2)!}\left(\frac{m+1}{2}+k\right)_{((l-1)/2)}q_{l+k}(x_0, \rho)\,P_k(\boldsymbol{\omega}); \quad l \text{ odd}. \quad (3.20b)$$

The expression (3.14) for the approximants then follows by substituting these expressions for $g_{l,k}(x)$ in (3.15).

4. Convergence Properties and Conclusions

We will prove uniform convergence of para-diagonal sequences of our approximants $f^j_{k,N}(x)$ to $f_k(x)$ in the domain $\mathscr{D}_m(\lambda, R)$ illustrated in Fig. 3.

For given $R > \lambda > 0$, we can choose $\frac{1}{4} > \delta_1 > 0$ so that this domain is contained in $\mathscr{D}_1 = \mathscr{D}_m(\delta_1^{1/2}, (4\delta_1)^{-1/4})$ for which we have defined our approximants.

Theorem 4.1. *For fixed* $j = -1, 0, 1, \ldots,$ *the approximants* $f^j_{k,N}(x)$ *converge uniformly to* $f_k(x)$ *as* $N \to \infty$ *for all* x *in* $\mathscr{D}_1$ *and hence* $\mathscr{D}_m(\delta, R)$ *when the* c_j*'s exist and satisfy condition* (1.6c).

Proof. We use the methods employed by Baker [2] to prove convergence of "Generalised Padé Approximants." We can write

$$f_k(x) - f^j_{k,N}(x) = \frac{1}{2\pi i} \int_C \frac{(1 + s\bar{x})}{[(1 + sx_0)^2 + s^2\rho^2]^{(m+1+2k)/2}}$$
$$\times [H(-1/s) - H_{N+j/N}(-1/s)] \frac{ds}{s} P_k(\mathbf{x}), \qquad (4.1)$$

where C is the contour in the complex s-plane illustrated in Fig. 2 with $\delta = \delta_1$.

Expanding the expression (3.13) for $H_{N+j/N}(z)$ in powers of z and using the defining property (1.3) for Padé approximants,

$$c_r = \sum_{n=1}^{N} \alpha_n \sigma_n^r + (-)^r \beta_r; \qquad r = 0, 1, \ldots, 2N + j \qquad (4.2)$$

with $\beta_r = 0$ for $r > j$. Then

$$H(z) - H_{N+j/N}(z) = \int_0^\infty \frac{(-uz)^{j+1}\, d\psi(u)}{(1 + zu)} + \sum_{l=0}^{j} (-z)^l c_l$$
$$- \left[\sum_{n=1}^{N} \frac{\alpha_n}{(1 + \sigma_n z)} + \sum_{l=0}^{j} \beta_l z^l \right]$$
$$= \int_0^\infty \frac{(-uz)^{j+1}\, d\psi(u)}{(1 + uz)} - \sum_{n=1}^{N} \frac{\alpha_n(-\sigma_n z)^{j+1}}{(1 + \sigma_n z)} \qquad (4.3)$$

and hence

$$H(-1/s) - H_{N+j/N}(-1/s) = \frac{1}{s^j} \left[\int_0^\infty \frac{u^{j+1}\, d\psi(u)}{(s - u)} - \sum_{n=1}^{N} \frac{\alpha_n \sigma_n^{j+1}}{(s - \sigma_n)} \right]. \qquad (4.4)$$

We now split the contour C into two parts, C_1 and C_2, with $\operatorname{Re} s \geqslant \mu_0 \geqslant \delta_1^{1/4}$ and $\operatorname{Re} s \leqslant \mu_0$, respectively. On C_1,

$$|s-u|, \qquad |s-\sigma_n| \geqslant \delta_1 \tag{4.5}$$

since u, σ_n are real non-negative. Therefore for $s \in C_1$,

$$|H(-1/s)-H_{N+j/N}(-1/s)| \leqslant \left|\frac{1}{s^j}\right| \frac{1}{\delta_1}\left[c_{j+1}+\sum_{n=1}^{N} \alpha_n \sigma_n^{j+1}\right], \tag{4.6}$$

where we have used the fact that the α_n's, σ_n's are postive. Using (4.2) with $r=j+1$,

$$|H(-1/s)-H_{N+j/N}(-1/s)| \leqslant \frac{2}{|s^j|} \frac{c_{j+1}}{\delta_1} \tag{4.7}$$

for $s \in C_1$. This bound with the bound (3.9b) on the functions $G_i(s, x_0, \rho)$ gives

$$\left|\frac{1}{2\pi i}\int_{C_1} G_i(s, x_0, \rho)[H(-1/s)-H_{N+j/N}(-1/s)] \frac{ds}{s}\right|$$
$$\leqslant \frac{2c_{j+1} b_i(\delta_1)}{\pi\delta_1}\int_{\mu_0}^{\infty} \mu^{-m-j-2k-1}\, d\mu; \qquad i=1, 2, \tag{4.8}$$

where $j=-1, 0, 1, 2, \ldots,$; $k=0, 1, 2, \ldots$. As we are considering the case with $m \geqslant 2$ "imaginary dimension," the RHS can be made smaller than a given $\varepsilon/2$ by choosing $\mu_0(\varepsilon, \delta_1)$ sufficiently large. This result also holds for $m=1$ if we restrict ourselves to $j \geqslant 0$.

For the contributions from C_2 we have the bounds,

$$\left|\frac{1}{2\pi i}\int_{C_2} G_i(s, x_0, \rho)\left[H(-1/s)-H_{N+j/N}(-1/s) \frac{ds}{s}\right]\right|$$
$$\leqslant \frac{a_i(\delta_1)(\mu_0+1)}{2\pi\delta_1}\int_{C_2} |H(-1/s)-H_{N+j/N}(-1/s)|\, |ds| \tag{4.9}$$

since $|s| \geqslant \delta_1$ on C_2 and (3.9a) has been used. Using the uniform convergence of $H_{N+j/N}(z)$ to $H(z)$ for any bounded closed domain of the complex z-plane not containing any point of the cut of $H(z)$, it follows that $N_0[\varepsilon, \delta_1, \mu_0(\varepsilon, \delta_1)]$ may be chosen so that the RHS of (4.9) is less than $\varepsilon/2$ for all $N \geqslant N_0$. Therefore for $i=1, 2$; $N \geqslant N_0[\varepsilon, \delta_1, \mu_0(\varepsilon, \delta_1)]$

$$\left|\frac{1}{2\pi i}\int_{C} G_i(s, x_0, \rho)[H(-1/s)-H_{N+j/N}(-1/s)] \frac{ds}{s}\right| < \varepsilon. \tag{4.10}$$

The difference $f_k(x) - f^j_{k,N})(x)$ is then made up of two components each of which tends to zero as $N \to \infty$, the convergence being uniform for all $x \in \mathscr{D}_1$ since N_0 is independent of x. The theorem then follows.

Using standard methods we have demonstrated that para-diagonal sequences of our approximants $\{f^j_{k,N}(x);\ N=0, 1, 2, ...\}$ converge to $f_k(x)$ uniformly for all x in the domain illustrated in Fig. 3 for $j = -1, 0, 1, ...$ We end this paper by demonstrating that our approximants given by (3.14) have a "power matching" property analogous to that used to define the Padé approximants to a function with a given Taylor's series.

Writing

$$\frac{(1+\sigma_n \bar{x})}{|1+\sigma_n x|^{m+1+2k}} = \frac{(\sigma_n^{-1}+\bar{x})}{\sigma_n^{m+2k}\,|\sigma_n^{-1}+x|^{m+1+2k}}$$

and using (2.13) with $t = 1/\sigma_n$, we have (formally)

$$\begin{aligned} f^j_{k,N}(x) = {} & \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\left[\frac{m+1}{2}+k\right]_l \left\{\sum_{n=1}^{N} \alpha_n \sigma_n^{2l}\right\} \\ & \times \{p_{2l+k}(x_0, \rho) - \sigma_n q_{2l+1+k}(x_0, \rho)\}\, P_k(\boldsymbol{\omega}) \\ & + \left\{\sum_{r=0}^{[j/2]} \beta_{2r} \frac{(-)^r}{r!}\left[\frac{m+1}{2}+k\right]_r p_{2r+k}(x_0, \rho)\right. \\ & \left. + \sum_{t=0}^{[(j-1)/2]} \beta_{2t+1} \frac{(-)^t}{t!}\left[\frac{m+1}{2}+k\right]_t q_{2t+k+1}(x_0, \rho)\right\} P_k(\boldsymbol{\omega}). \end{aligned} \tag{4.11}$$

This may be compared term by term with the expansion

$$\begin{aligned} f_k(x) = {} & \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\left[\frac{m+1}{2}+k\right]_l \\ & \times [p_{2l+k}(x_0, \rho)\, c_{2l} - q_{2l+k+1}(x_0, \rho)\, c_{2l+1}]\, P_k(\boldsymbol{\omega}). \end{aligned} \tag{4.12}$$

Using the identities (4.2) it is easy to check that

$$f_k(x) - f^j_{k,N}(x) = O(p_{2N+j+k+2}, q_{2N+j+k+1}); \qquad j \text{ even} \tag{4.13a}$$

$$= O(p_{2N+j+k+1}, q_{2N+j+k+2}); \qquad j \text{ odd}. \tag{4.13b}$$

This is the "power matching" property of our approximants.

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